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LETTER TO THE EDITOR

Finite size effects in conformal field theories and non-local operators in one-dimensional quantum systems

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Abstract. We generalise the well known connection between critical exponents and finite size effects in conformal field theories to non-local operators and find the long-wave asymptotics of vacuum expectation values of some non-local operators in one-dimensional quantum systems.

Recently the method to find out the long-wave asymptotics of correlation functions of various local operators in one-dimensional quantum systems has been proposed [1–5]. This method is based on the investigation of finite size effects in conformal field theories [6, 7]. The idea is to use the fact that at zero temperature one-dimensional models of quantum field theory are conformally invariant at large distances.

More precisely, the dimensions Δ, $\bar{\Delta}$ of primary fields $\phi_{\Delta,\bar{\Delta}}$ [8] are connected with the energies E_L^ϕ of the lowest excitations $|\phi\rangle$ such that $\langle \mathrm{vac}|\phi|\phi\rangle \neq 0$, by the following relation [6]:

$$E_L^\phi - E_L^{\mathrm{vac}} = 2\pi v h_\phi / L. \tag{1}$$

Here $h_\phi \equiv \Delta_\phi + \bar{\Delta}_\phi$ is the scaling dimension of the operator ϕ, L is the length of the system, v is the sound-wave velocity in the system (the group velocity at the Fermi surface) and E_L^{vac} is the energy of the ground state. The long-wave asymptotics of the $\phi_{\Delta,\bar{\Delta}}$ field correlator (for simplicity, we consider only the equal time correlators throughout this paper) has the form [5, 6]:

$$\langle \phi(x)\phi(0)\rangle \sim \cos(P^\phi x) x^{-2h_\phi} \tag{2}$$

where P^ϕ is the momentum of the state $|\phi\rangle$. Non-zero P^ϕ implies a gap in the momentum spectrum.

Similarly, the central charge c of the corresponding Virasoro algebra is related to the L^{-1} correction to the ground state energy [7]. For the simplest one-component systems (as one-dimensional Bose or Fermi gas or $s=\frac{1}{2}$ Heisenberg antiferromagnet etc) one obtains $c=1$ [1, 2, 4, 5, 7]. It follows that all these theories belong to the universality class of the one-component Gaussian model [9]. The dimensions of local primary operators ϕ_{nm} in the Gaussian model depend on the only continuous parameter R (which coincides with compactification radius in string theory) and have the form

$$h_\phi = n^2/R^2 + m^2R^2/4 \qquad n, m \text{ are integers.} \tag{3}$$

The value of R depends on characteristics of the model [5]: $R^2 = 8\pi N/vL$, where N is the number of particles in the system (or the number of reversed spins in the case of an antiferromagnet). The thermodynamic limit corresponds to $L\to\infty$, $N\to\infty$, $N/L=\rho=\text{constant}$.

We will show that a simple extension of the above method allows one to find out the correlator of some non-local operators in one-dimensional systems.

To begin with, let us consider the XXZ Heisenberg antiferromagnet with the standard Hamiltonian

$$\hat{H}_1=-\frac{1}{2}\sum_{i=1}^{L}(\sigma_i^1\sigma_{i+1}^1+\sigma_i^2\sigma_{i+1}^2-\cos(\gamma)\sigma_i^3\sigma_{i+1}^3)\qquad 0\leq\gamma<\pi \tag{4}$$

and periodic boundary conditions.

It is interesting [10, 11] to find out the vacuum expectation values of the following non-local operators:

$$S_{xy}\equiv\prod_{j=x}^{y}\sigma_j^3=\exp\{\mathrm{i}\pi q(x,y)\} \tag{5}$$

$q(x, y)$ being the operator of the number of reversed spins at the sites on the way from x to y, and

$$T_{xy}=P_{x(x+1)}P_{(x+1)(x+2)}\ldots P_{(y-1)y}P_{yx} \tag{6}$$

where $P_{xy}\equiv\frac{1}{2}(1+\sigma_x\sigma_y)$ denotes the spin exchange operator at the sites x and y. The operator T_{xy} acts by a cyclic permutation of the sites along the segment $[x, y]$: $x\to x+1, x+1\to x+2,\ldots,y-1\to y, y\to x$. Due to the translational invariance, $\langle\text{vac}|S_{xy}|\text{vac}\rangle$ and $\langle\text{vac}|T_{xy}|\text{vac}\rangle$ depend only on $|x-y|$. Here $|\text{vac}\rangle$ denotes the ground state of the antiferromagnet with periodic boundary conditions.

Let us consider an extended Hilbert space containing all states of the spin chain with different number of sites and different boundary conditions simultaneously. We introduce the following operators acting in this extended space: the operator a_{xs}^+ creating a new site of spin s ($s=\pm1$) between the sites x and $x+1$ of the initial chain; the operator b_{xs} annihilating the site x (of spin s_x) in the case of $s_x=s$ and giving zero if $s_x=-2$. Thus, a_{xs}^+ acts from the sector of the extended Hilbert space corresponding to the chain with L sites to the sector corresponding to $L+1$ sites. Quite similarly, b_{xs} changes the number of sites from L to $L-1$. We also introduce the operator

$$S_x\equiv\prod_{j=x}^{L}\sigma_j^3 \tag{7}$$

which connects the sectors corresponding to periodic and antiperiodic boundary conditions. Clearly,

$$T_{xy}=\sum_s a_{xs}^+b_{ys} \tag{8a}$$

$$S_{xy}=S_xS_y. \tag{8b}$$

Let us denote by $|L,+\rangle$ ($|L,-\rangle$) the ground state of the spin chain with L sites and periodic (antiperiodic) boundary conditions (in the thermodynamic limit these states coincide with $|\text{vac}\rangle$, but we are interested in finite size corrections $\sim L^{-1}$). One can easily check that

$$\langle L-1,+|b_{xs}|L,+1\rangle\neq0\qquad\langle L+1,+|a_{xs}^+|L,+\rangle\neq0\qquad\langle L,-|S_x|L,+\rangle\neq0. \tag{9}$$

This means that in order to find out the asymptotics at $|x-y|\gg 1$ of $\langle\text{vac}|T_{xy}|\text{vac}\rangle = 2\langle a^+_{x(+1)}b_{y(+1)}\rangle$ (obviously, $\langle a^+_{x(+1)}b_{y(+1)}\rangle = \langle a^+_{x(-1)}b_{y(-1)}\rangle$) and $\langle\text{vac}|S_{xy}|\text{vac}\rangle = \langle\text{vac}|S_xS_y|\text{vac}\rangle$ we can use the formula (2) with the scaling dimensions being determined from the relation (1). In other words, one may consider the states $|L,-\rangle$ and $|L+1,+\rangle$ as 'excitations' over the 'true' ground state $|L,+\rangle$. It is sufficient to calculate the energies of these excitations up to the first order in L^{-1}. One can do this using the well known exact solution. Namely, the straightforward calculations based on the Bethe equations in large L limit give

$$\langle\text{vac}|T_{xy}|\text{vac}\rangle \sim |x-y|^{-\mu} \tag{10a}$$

$$\langle\text{vac}|\exp\{\mathrm{i}\pi q(x,y)\}|\text{vac}\rangle \sim \cos[\pi(x-y)/2]|x-y|^{-\lambda} \tag{10b}$$

where $\mu = R^{-2}/2$, $\lambda = R^2/8$ and $R^2 = 2\pi(\pi-\gamma)^{-1}$ coincides with the inverse scaling dimension of the operators σ^1, σ^2 [1, 4]. Let us note that similar results for the energy shifts were obtained in [1] by numerical estimations.

Certainly, there are many other 'lowest' excitations over the ground state and their energies give the full spectrum of scaling dimensions of the primary fields in the extended (non-local) Gaussian model [12]. It turns out that this spectrum is described by the same formula (3) but with half-integer n and m. In particular, $b_{\pm 1}$ can be identified with the non-local operators $\phi_{\pm 1/2,0}$ in the extended Gaussian model [9].

Note that in the case of the XXX antiferromagnet ($\gamma = 0$ in (4)) the operator S_X has the scaling dimension $\frac{1}{8}$ and coincides with the spin field in conformal field theory [13].

The same approach applies [5] to the continuous model of a spinless Bose or Fermi gas with the Hamiltonian

$$\hat{H}_2 = \int_0^L \mathrm{d}x\, \partial_x\psi^*(x)\partial_x\psi(x) + \tfrac{1}{2}g \iint_0^L \mathrm{d}x\,\mathrm{d}y\, \psi^*(x)\psi^*(y)V(x-y)\psi(x)\psi(y) \tag{11}$$

where $g>0$ and $V(x)$ is some potential of pairwise interaction (repulsion) of quite general form. It is hard to formulate the necessary and sufficient conditions for the absence of the gap, in terms of the potential $V(x)$. We are convinced, however, that the class of potentials in (11) leading to the gapless sound-type spectrum is quite representative.

Indeed, the cases of small and large constants g in (1) have been studied in the literature. In [14], where the case of small g in Fermi systems was studied perturbatively, it was concluded that the correlators behave as a power law and no gap emerges. The case of long-range $V(x)$ and large g (strong repulsion) was considered in [10, 15]. In this case, the absence of the gap becomes quite obvious. Indeed, independently from $V(x)$, the particles form a regular 'dynamical lattice' (the so-called Wigner crystal) with sound-type low-energy excitations. On the other hand, there exist several exactly solvable models which possess a gapless spectrum. These are, for example, the model with delta-shape potential and that with the potential $V(x) = x^{-2}$ (the Sutherland model [16]). Evidently, the gap will not appear when performing small deformations of the Sutherland potential which do not entail a qualitative rearrangement of the ground state.

Now the analogue of the operator S_{xy} in the systems with Hamiltonian (11) is given by just the same expression as on the right-hand side of the formula (5) where now $q(x,y)$ is the operator of the number of particles on the segment $[x,y]$. We have (at

$|x-y| \gg L/N$)

$$\langle \mathrm{vac}|S_{xy}|\mathrm{vac}\rangle \sim \cos(\pi\rho|x-y|)|x-y|^{-1/4\theta} \tag{12}$$

where $\theta = 2R^{-2} = v(4\pi\rho)^{-1}$ is the critical exponent of the correlator of the bosonic fields ψ [5, 10], $\rho = N/L$ denotes the density of particles and v is the same quantity as in (1).

The operator S accomplishes the Jordan–Wigner transformation from the bosonic operators ψ_{B} to the fermionic ones ψ_{F}:

$$\begin{aligned} \psi_{\mathrm{F}}(x) &= \psi_{\mathrm{B}}(x)S_{xL} \\ \psi_{\mathrm{F}}^*(x) &= S_{xL}^*\psi_{\mathrm{B}}^*(x). \end{aligned} \tag{13}$$

This fact allows one to identify ψ_{F} with the non-local operator $\phi_{1,1/2}$ in the extended Gaussian model and calculate the field correlator in the fermionic model (11):

$$\langle \mathrm{vac}|\psi_{\mathrm{F}}^*(x)\psi_{\mathrm{F}}(y)|\mathrm{vac}\rangle \sim \cos(\pi\rho|x-y|)|x-y|^{-\theta-1/(4\theta)}. \tag{14}$$

In the general case the spectrum of scaling dimensions in the Fermi gas (11) is described again by the formula (3) with the following conditions: if n is even, m is an integer and if n is odd, m is half-odd integer.

A detailed version of this letter will be published elsewhere.

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